ON THE SOLUTION OF A DEGENERATE VARIATIONAL PROBLEM AND THE OPTIMUM CLIMB OF A COSMIC [SPACE] ROCKET

(O RESHENII ODNOI VYROZHDENNOI VARIATSIONNOI ZADACHI I OPTIMAL'NOM POD"EME KOSMICHESKOI RAKETY)

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V.A. EGOROV
(Moscow)

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The paper contains a solution of Mayer's problem for Pfaff's equation with one free function, and is applicable to the selection of the climbing trajectory of a rocket to a given altitude with maximum speed. The indicated problem is formulated in Section 1, and it is shown that the problems of the rocket flight without induced drag and of its motion, with zero angle of attack, on a rigid, ideally smooth track, are reducible to it. It is proved that the formulated problem is a degenerate one. In Section 2 the variation is investigated and a general solution of the problem is given, as well as basic special cases. In Section 3 the application of the general solution to the problem of rocket climb yields the optimum trajectory. The case of the motion on a launching track [ramp] is considered separately.

1. A series of problems concerning the determination of optimum regimes [programmes] of the motion of the center of gravity of a rocket in a resisting medium are reducible to the following problem of Mayer for Pfaff's equation.

In the plane of the variables x_2, x_3 there is given a simply connected region σ_1 with a closed boundary σ_1^0. There is to find, within σ_1, a piecewise smooth curve without multiple points $\gamma_1\{x_2 = x_2(\tau),\ x_3 = x_3(\tau),\ 0 \leqslant \tau \leqslant 1\}$ with prescribed end points $A_1 \varepsilon\, \sigma_1^0$ and $B_1\ \varepsilon\ \sigma_1^0$ such that the value $x_1(1)$ of the function $x_1(\tau)$, determined along γ_1 by the equation

$$X_1(x_1, x_2, x_3)\,dx_1 + X_2(x_1, x_2, x_3)\,dx_2 + X_3(x_1, x_2, x_3)\,dx_3 = 0 \qquad (1.1)$$

and the initial value $x_1(0) = x_{1A}$, be a maximum, and that the direction of the tangent to the space curve $\gamma\{x_i = x_i(\tau),\ 0 \leqslant \tau \leqslant 1,\ i = 1,2,3\}$ at each of its points belong to a prescribed space bundle [pencil] $\omega'(x_1, x_2, x_3)$ of admissible directions. Thereby, the functions

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$X_i(x_1, x_2, x_3)$ ($i = 1,2,3$) are considered continuous, together with their partial derivatives and the function $X_1 \neq 0$ for the values of the variables considered.

It is not difficult to show that several other classes or problems are reducible to the one stated above. The first class of problems concerns the optimum expenditure of fuel in rectilinear motions of rockets such as, for example, the interrelated problems of reaching maximum altitude with given fuel consumption [1], of reaching a given altitude with least expenditure of fuel [2], and of reaching maximum speed at prescribed altitude.

Secondly, the general problem is reducible to the problems concerning optimum trajectories in the vertical plane for winged rockets with a prescribed thrust regime, but without taking into account, in the equations of motion, the second and higher powers of the angles of attack, in particular without taking into account induced drag. These are problems which have to do with such a determination of trajectories, that the expended fuel (or the flight duration) be a minimum for prescribed terminal values of altitude and speed [4], or that for prescribed flight duration the speed be a maximum at prescribed altitude, or that the flight altitude be a maximum under the condition that the flight time and the final speed are prescribed.

Thirdly, problems on the motion of a rocket with zero angle of attack on a rigid, ideally smooth track in which the shape of the track curve is to be found are also reducible to the general problem. These are problems on the climb of a cosmic rocket on the launching track (or pad) in the absence of friction. The boundary parameters are in this case the parameters at the end of the track.

To prove that the problems of the second class are reducible to the general problem, we consider the equation of motion of a rocket with given thrust regime, without taking into account the second and higher powers of the angle of attack (Fig. 1).

$$\frac{dv}{dt} = p - \frac{g}{\mu G_0} X(v, y) - g \sin \theta \tag{1.2}$$

$$v \frac{d\theta}{dt} = \alpha p + \frac{g}{\mu G_0} \alpha Y_1(v, y) - g \cos \theta \tag{1.3}$$

$$\frac{dy}{dt} = v \sin \theta \tag{1.4}$$

Here α is the angle of attack, θ is the angle between the tangent to the trajectory and the horizon, p is the ratio of the thrust P to the mass $\mu(t)\, G_0/g$, v is the speed of motion, y is the altitude of flight, $X(v,y)$ is the frontal resistance and $Y(v,y) = \alpha Y_1(v,y)$ is the lift. Equations (1.2) to (1.4) contain v, θ, y and α as unknown functions of time t. If one of them is given, the others may be determined from the above equations. From the formal point of view, all these functions are mathematically on equal footing, and any one among them may be considered as being free. However, physically free is only α, because the angle $\alpha(t)$ may be altered directly during flight in accordance with the

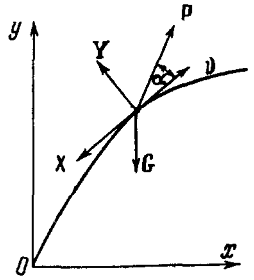

Fig. 1.

prescribed program, while the other functions may be influenced only through $\alpha(t)$.

We should note now that α does not enter into equation (1.2) and θ may be eliminated from (1.2) with the aid of (1.4). After this elimination we obtain instead of (1.2) an equation of the type (1.1) which contains only v, t, y:

$$\mu v\, dv + v(\varphi - \mu p)\, dt + \mu g\, dy = 0, \qquad \varphi = \frac{gX}{G_0} \tag{1.5}$$

Equation (1.5) may be separated from the others and variational problems of the second type may be formulated independently of the other equations. In fact, these problems concern merely the variables v, t, y and if one of the functions $v(t), y(t)$ is considered as free, and is prescribed in an arbitrary fashion, then the other may be determined from (1.5), without use of the other equations. Subsequently, if it should be necessary, $\theta(t)$ and $\alpha(t)$ may be found from (1.4) and (1.3).

We also note that the condition $|dy| \leqslant v dt$, which results from (1.4), is not taken into account at all in (1.5) and therefore this should be remembered in the formulation of the problem. This means that in the space v, t, y the motion from the point (v, t, y) may continue only within a local space angle ω' [formed by two planes] with the edge parallel to the v-axis and with sides inclined to the plane y=const through an angle $\pm$ arc tan v, where the plus and minus signs correspond to climb and fall along the vertical, respectively.

Now, the problem on the climb to a prescribed altitude with maximum velocity, for instance, may be formulated as follows.

A function $y(t)$ is to be found in such a manner that the value $v(T)$

of the function $v(t)$ determined by the initial value $v(t_0) = v_0$ and the equation (1.4) should be a maximum and that the conditions be satisfied

$$y = y_0 \quad \text{for } t = t_0, \qquad y = y_k \quad \text{for } t = T \tag{1.6}$$

$$y_0 \leqslant y \leqslant y_k \qquad \text{for } t_0 \leqslant t \leqslant T \tag{1.7}$$

$$dt > 0, \qquad -v\,dt \leqslant dy \leqslant v\,dt \tag{1.8}$$

Comparing this formulation with the general formulation for equation (1.1), we see that in fact this problem is reducible to the general one. Since other problems of the second type for equation (1.5) may be formulated in an analogous manner, their reducibility to the general problem is proved.

To prove the reducibility of problems of the third type, we will show that the totality of motions corresponding to all possible values of functions $\alpha(t)$ corresponds to the totality of motions with zero angle of attack on all possible ideally smooth and absolutely rigid tracks of

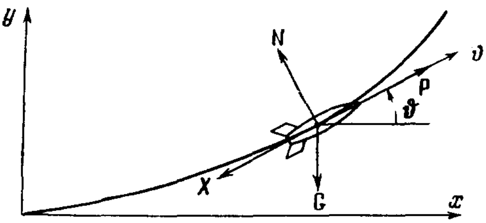

Fig. 2.

constraint (Fig.2). In fact, the first and the third equation of motion coincide in this case with (1.2) and (1.4), and the second equation of this motion

$$v\frac{d\theta}{dt} = N - g\cos\theta$$

is identical with (1.3), if the reaction of constraint is

$$N = \alpha\left(p + \frac{g}{\mu G_0} Y_1\right)$$

For actual motion ($v > 0$), this condition may always be satisfied by choosing a track of suitable curvature if the function $\alpha(t)$ is given, or by choosing the function $\alpha(t)$ if the motion on the trajectory of constraint is given. The coincidence of equations indicates a coincidence of motions with like initial conditions and in particular the coincidence of trajectories. From this it follows, by the way, that in the problems of both types the normal forces may be of arbitrary magnitude and may be even infinite, and that the trajectories, correspondingly, may have corners. This last circumstance is explained by the fact that the magnitude of normal forces is not connected in the problems considered, in contrast to real problems, to the increase of losses due to resistance.

We will show now that the general problem formulated above is a degenerate one. To this end, we express the variation δx_{1B} through δx_2 and δx_3, i.e. through the variation of the curve γ_1. Integrating in general terms the linear variational equation associated with equation (1.1) along the solution of this equation, i.e. the curve γ, which possesses an admissible curve γ_1 of its own projection and which satisfies the condition $x_1(0) = x_{1A}$, we obtain an expression of the variation δx_1 in terms of $\delta\dot{x}_2$, $\delta\dot{x}_3$, δx_2, δx_3, $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ and the functions of x_1, x_2, x_3 (a dot indicates differentiation with respect to τ). Substituting $\tau = 1$, we obtain $\delta x_{1B} = \delta x_1(1)$. Transforming terms with $\delta\dot{x}_2$, $\delta\dot{x}_3$ through integration by parts, using equation (1.1) and the end conditions at A_1 and B_1, we obtain the sought expression for the variation of the functional

$$\delta x_{1B} = \int_{(\gamma)} k(M)\,\Phi'(x_1, x_2, x_3)\,(\delta x_2 dx_3 - \delta x_3 dx_2) \tag{1.9}$$

where

$$k(M) = \frac{1}{X_1^2}\exp\int_B^M \frac{1}{X_1}\left(\frac{\partial X_1}{dx_1}dx_1 + \frac{\partial X_2}{\partial x_1}dx_2 + \frac{\partial X_3}{\partial x_1}dx_3\right)$$

$$\Phi'(x_1, x_2, x_3) = X_1\left(\frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2}\right) + X_2\left(\frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3}\right) + X_3\left(\frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_3}\right) \tag{1.10}$$

We note that it is valid everywhere because of the condition $X_1 \neq 0$. Since the integrand does not contain $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3$, we have in fact a special degenerate case of the variational problem [3]. The corresponding Euler equation, as is known, may have no solution in the region σ_1. Therefore, the solution of the problem should be sought by the method of direct investigation of the variation.

2. It is not difficult to convince oneself, through direct verification, that the expression of the variation (1.9) may be written down in terms of a line integral, with weight $k(M)$ (where M is an arbitrary point of the curve γ), as a function of the scalar product of the displacement vector $\mathbf{d}_1$, transformed by a matrix H, and the vector $\boldsymbol{\delta}_1$ of the variation of the point on the curve γ_1, i.e.

$$\delta x_{1B} = \int_{(\gamma)} k(M)\, \mathrm{H}\mathbf{d}_1 \cdot \boldsymbol{\delta} \tag{2.1}$$

$$\mathrm{H} = \begin{Vmatrix} 0 & \Phi' \\ -\Phi' & 0 \end{Vmatrix}, \qquad \mathbf{d}_1 = \begin{Vmatrix} dx_2 \\ dx_3 \end{Vmatrix}, \qquad \boldsymbol{\delta}_1 = \begin{Vmatrix} \delta x_2 \\ \delta x_3 \end{Vmatrix} \tag{2.2}$$

Subscript 1, as before, will indicate the projection on the plane x_2, x_3 of corresponding quantities in space.

We note that the left-hand side of (1.1) represents the scalar product $\mathbf{X} \cdot \mathbf{d}$ where $\mathbf{X} = (X_1, X_2, X_3)$, and $\mathbf{d} = (dx_1, dx_2, dx_3)$, such that, if relations (1.1) are satisfied, the vector $\mathbf{d}$ is in the plane π, which is orthogonal to $\mathbf{X}$. Let ω (M) be the local angle determined by π and the space angle $\omega'(M)$, and its projection on the plane x_2, x_3 be $\omega_1(M)$. Obviously, the admissible, piecewise smooth curves with ends A_1 and B_1 in the region σ_1 will be then those whose tangents at each point M_1 belong to the bundle $\omega_1(M)$. Let $\Gamma'(P)$ be a curve originating at point P, which satisfies equation (1.1) and is in contact with the left boundaries of the angles $\omega(M)$, where $M \in \Gamma'(P)$. We shall çall then the projection of this curve $\Gamma'(P)$ the left internal boundary, and the analogous curve for the right boundaries of the angles $\omega(M)$ will be called the right internal boundary Γ_1'' (P).

The matrix H is skew symmetric and rotates the vector $\mathbf{d}_1(M)$ through an angle $\pi/2$ in the clockwise sense, if the quantity $\Phi'(M) > 0$, and in the counterclockwise sense if $\Phi'(M) < 0$ and makes it zero if $\Phi'(M) = 0$. Let us consider the initial portion of the admissible curve γ_1. It is situated within the angle $\omega_1(A)$. To fix the ideas, let $\Phi' > 0$ in a given point A. Then, since the function $\Phi'(M)$ is continuous, it will be positive in points within a sufficiently small portion AM of the curve γ.

The vector $\mathrm{H}\mathbf{d}_1$ on the corresponding portion A_1M_1 (Fig. 3) is directed to the right of the curve γ_1. Using the fact that the portion A_1M_1 is situated within the angle $\omega_1(A)$ and the angles in the neighboring points, we shift it towards the vector $\mathrm{H}\mathbf{d}_1$, i.e. we direct the vectors $\boldsymbol{\delta}_1$ to the

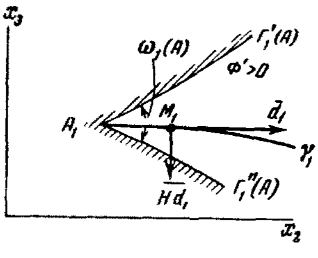

Fig. 3.

right in the portion A_1M_1, and in the remaining points of the curve γ_1 we put $\boldsymbol{\delta}_1$ equal to zero. Then the angle between the vectors H $\mathbf{d}_1$ and $\boldsymbol{\delta}_1$ will be acute, $\mathrm{H}\mathbf{d}_1 \cdot \boldsymbol{\delta}_1 > 0$ in the whole portion A_1M_1 and

$$\delta x_{1B} = \int\limits_{AM} k\,\mathrm{H}\mathbf{d}_1 \cdot \boldsymbol{\delta}_1 > 0$$

It follows, that it is possible to "improve" the curve γ_1 by means of such a variation.

The improvement of the initial portion of γ_1 through the described variation may be carried out as long as it does not pass along the internal boundary $\Gamma_1''(A)$. Further variation is not possible: the curve γ_1 will cease to be an admissible one. Inasmuch as the given reasoning may be applied also for the neighboring portions of the curve γ_1 as well, for which $\Phi' > 0$, the sought solution of the problem Γ_1, beginning at the point A_1, should pass along the internal boundary $\Gamma_1''(A)$. (Analogously, if $\Phi'(A) < 0$, the solution should pass along the left internal boundary).

It may happen, however, that $\Phi'(C) = 0$ for some point $C_1 \in \Gamma_1''$ and $\Phi' < 0$ for points of this boundary further along, i.e. the curve Γ_1'' will encounter a surface $\Phi'(x_1, x_2, x_3) = 0$. We construct then the angle $\omega_1(C)$. If the projection ϕ_1 of the curve ϕ, determined by the point C and equations $\Phi'(x_1, x_2, x_3) = 0$ and (1.1), passes outside the angle $\omega_1(C)$, then $\Phi' < 0$ everywhere within the angle $\omega_1(C)$ and the solution Γ_1 from the point C_1 on, should pass then along the left boundary $\Gamma_1'(C)$, being broken at the point C_1 through an angle $\omega_1(C)$.

Let now the curve ϕ_1 pass within the angle $\omega_1(C)$, that is to the left of the curve Γ_1'', along which the motion took place so far. If a small portion C_1M_1 of admissible continuation of γ_1 from the point C_1 on, is taken to the right of the curve ϕ_1, that is in the region $\Phi' < 0$, then the vectors $\mathrm{H}\mathbf{d}_1$ will be directed to the left, to the curve ϕ_1; if on the other hand the portion C_1M_1 is taken to the left of the curve ϕ_1, then the vectors $\mathrm{H}\mathbf{d}_1$ will be directed to the right, that is again towards the curve ϕ_1. This means that shifting the portion C_1M_1 towards the curve ϕ_1, we will obtain necessarily in both cases $\delta x_{1B} > 0$, and the portion $C_1M_1 \in \phi_1$ cannot be improved any further. Therefore, the solution Γ_1 will again be broken at the point C_1 and pass from it along the curve ϕ_1 as long as it is an admissible one.

It may also happen, however, that in some point N_1 the curve ϕ_1 will touch one of the boundaries of the angle $\omega_1(N)$ and then ceases to be an admissible one (its points M_1, following the point N_1, will be outside the angle $\omega_1(M)$). Then it can be shown by variation, that from the point N_1 on, the solution Γ_1 should pass through that corresponding point N of the internal boundary, which was touched by the curve ϕ_1

at the point N_1. If, however, the curve ϕ_1 remains an admissible one after the contact, the solution Γ_1 continues to pass along this curve.

Furthermore, the exceptional case is possible, when $\Phi'(A) = 0$. Then two alternatives present themselves. The first occurs when the point A is analogous to the point C, which we already discussed. In the second case, which is essentially different, the curve ϕ_1 is admissible, but $\Phi' > 0$ is to the right of ϕ_1 and $\Phi' < 0$ is to the left of ϕ_1, such that the vectors $H\mathbf{d}_1$ are directed not towards the curve ϕ_1 but away from it. It can be shown then that the curve ϕ_1 corresponds not to a maximum but to a minimum x_{1B} and that, under the conditions of the absence of multiple points [of multivaluedness] on admissible curves, the maximum will correspond to one of the internal boundaries for the point A (which one in particular is cleared up by direct verification).

It is obvious that the solution Γ_1 may pass only through those points M_1 of the curves ϕ_1, Γ_1', Γ_1'', for which admissible end conditions of motion exist, i.e. the curves M_1B_1. But, sooner or later, the solution on one of these curves ϕ_1, Γ_1' or Γ_1'' will encounter a point D_1 such that the point B_1 will be on one of the internal boundaries $\Gamma_1'(D)$ or $\Gamma_1''(D)$ and for points M_1, following D_1 the point B_1 will be beyond this boundary, (Fig. 4). As a result, the curves M_1B_1 will cease to be admissible ones. In this case, with the aid of variation, it is possible to show that the solution Γ_1 must pass along a portion D_1B_1 of those boundaries $\Gamma_1'(D)$ or $\Gamma_1''(D)$, on which the point B_1 happened to be, and has to terminate at the point B_1. (In a special case, the portion D_1B_1 may be equal to zero).

From the solution obtained it may be seen that is can consist only of portions of the curves $\phi, \Gamma_1', \Gamma_1''$, following each other in a definite sequence. Inasmuch as all the conceivable cases of passing from one portion to the next were considered, the solution obtained is a general

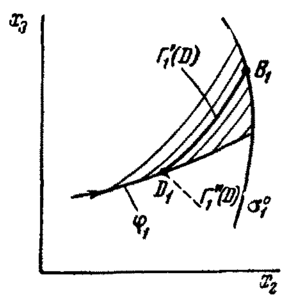

Fig. 4.

one. It is obvious that it always exists, provided at least one permissible curve A_1B_1 exists.

The maximum problem is solved analogously to the minimum problem.

In this manner, having found the surface $\Phi(x_1, x_2, x_3) = 0$, and departing from point A, we always can find the solution Γ_1 with the aid of the method described above.

Remarks. 1. It is easily observed that the method of solution described above is also applicable if the point B_1 is not fixed, but a curve $\beta\{x_2(\tau), x_3(\tau)\}$ is given, on which it must be chosen in such a manner that the quantity x_{1B} be a maximum.

Suppose that for an arbitrarily fixed point $\beta_1 \epsilon \beta$ some best curve γ is obtained. Through variation of the point β_1 we obtain a variation in form of a sum of integral and non-integral terms, whereby in varying the end β_1, obviously, only the part D_1B_1 of the solution Γ_1 will be changed. The solution of the problem with a moving end, obviously, is determined by the condition of mutual compensation of the integral and non-integral terms for an infinitely small displacement of the end.

2. The equation $\Phi'(x_1, x_2, x_3) = 0$, as it follows from (2.1), is an analog of Euler equation for the degenerate problem considered. Therefore, the curves ϕ_1 satisfying the equations $\Phi' = 0$ and (1.1) represent the essence of the extremals analogy. This analogy is also confirmed by the fact that they are the same for the corresponding degenerate problems. In fact, in the variation expression for the corresponding problem of the extremum x_2, the expression Φ'' analogous to Φ', must be obviously, obtained from (1.10) only by an interchange of subscripts one and two. But such an interchange, as a direct verification shows, yields an expression Φ'' coinciding with Φ' or differing only by a sign. Inasmuch as equation (1.1) is the same for both problems, the extremals analogies coincide also. They differ from the usual extremals (solutions of usual Euler equations) by the fact that their projections, in general, do not pass through given points A_1 and B_1, and their values are not sufficient for the construction of a solution of the degenerate variational problem.

3. Let us study the case $\Phi' \equiv 0$. If the condition is considered concerning the complete integrability of Pfaff's equation (1.1), that is the condition of the existence of its integral in the form $F(x_1 x_2 x_3) = c$ (where $c =$ const), then this condition turns out to be precisely the equality $\Phi' = 0$. From the relation $F = c$, x_1 may be expressed through x_2 and x_3, whereby for various values of the constant c we obtain different surfaces of the family $x_1 = x_1(x_2, x_3, c)$. Obviously, fully determined surfaces $x_1 = x_1(x_2, x_3, c_A)$ and $x_1 = x_1(x_2, x_3, c_B)$ are passing through points A and B. In this case the coordinates x_{2B}, x_{3B} of the point B_1 determine uniquely the quantity x_{1B}, independently of the form of the admissible curve γ_1. (This is confirmed also by the identical vanishing of the variation δx_{1B}). As a result, all the curves are seen

to be equivalent to each other and the problem of the calculus of variations loses its meaning.

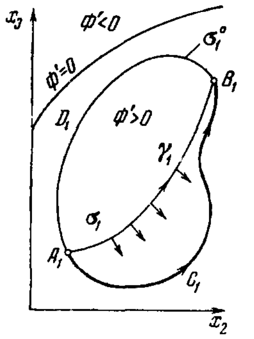

Fig. 5.

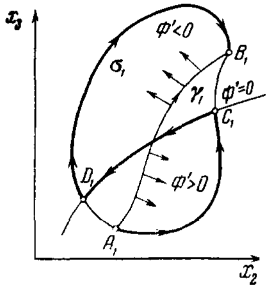

Fig. 6.

4. Let us apply the stated method to the case when the angle $\omega_1 \equiv 2\pi$ and the function Φ' depends only on x_2 and x_3. In this case, it is not necessary to consider three-dimensional geometrical figures and we confine ourselves merely to the discussion of their projections on the plane x_2, x_3. The factor $k(M)$, entering the expression for the variation, which depends essentially on the position of the curve γ in space, does not preclude such a simplified discussion because of its determined sign. The simplicity of the investigation is based here on the fact, that the sign of the function $\Phi'(x_2, x_3)$ at the point M_1 of the curve γ_1 does not depend on the quantity x_1, that is on the preceding behavior of this curve. Therefore, the curve $\Phi' = 0$ on the plane x_2, x_3 may be indicated beforehand and it will be the same for all admissible curves γ_1. If the curve $\Phi' = 0$ intersects a given domain σ_1, it subdivides it into subdomains $\Phi' > 0$ and $\Phi' < 0$. If, however, it does not intersect σ_1, the sign of Φ' within σ_1 does not change. Let us consider this last case first, since it is the simpler one. To fix the ideas, we assume $\Phi' > 0$ within σ_1, (Fig.5). Then, as it follows from Section 2, any curve γ_1, connecting the points A_1 and B_1 should be varied to the right in order to obtain a maximum. Improving γ_1 through such a variation, we arrive at a curve, coinciding with the portion $A_1C_1B_1$ of the boundary $\sigma_1{}^0$. This curve represents the solution of the maximum problem on x_{1B}, because it can no longer be improved. Analogously, the solution of the minimum problem is represented by the portion $A_1D_1B_1$ of the boundary $\sigma_1{}^0$.

If $\Phi' < 0$ within σ_1 then $A_1C_1B_1$ corresponds to a minimum, and $A_1D_1B_1$ to a maximum.

To obtain the maximum $\dot{x}_{1B}$ in the case when the curve $\Phi' = 0$ intersects the domain σ_1, through variation of an arbitrary admissible curve γ_1 to the right in the sub-domain $\Phi' > 0$ and to the left in the sub-domain $\Phi' < 0$ (as is indicated by arrows in Fig.6) we find, that the solution is represented by the curve $A_1C_1D_1B_1$ (it is impossible to improve this curve further). Analogously, the solution of the minimum problem x_{1B} is represented by the curve $A_1D_1C_1B_1$.

5. Let us consider now that special case of the foregoing, when the functions X_2/X_1 and X_3/X_1 do not depend on x_1. Then, obviously, we can simply say that in equation (1.1) $X_1 = 1$, while x_2 and x_3 do not depend on x_1. In this case, through integration along the curve γ_1, it is possible to obtain not only the variation

$$\delta x_{1B} = \int\limits_{\gamma_1} \left(\frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2}\right)(\delta x_2\, dx_3 - \delta x_3\, dx_2)$$

but also the functional itself

$$x_{1B} = x_{1A} + \int\limits_{\gamma_1} [M(x_2, x_3)\, dx_2 + N(x_2, x_3)\, dx_3]$$

where $M = -X_2$, $N = -X_3$. It differs only through the constant additive term x_{1A} from the degenerate functional of the most simple problem [3]. Considering the domain σ_1 as being given by a portion of the strip $a < x_2 < b$ within non-intersecting curves $x_3 = f_1(x_2)$, $x_3 = f_2(x_2)$, (Fig.7), and applying the result obtained above, the solutions for the

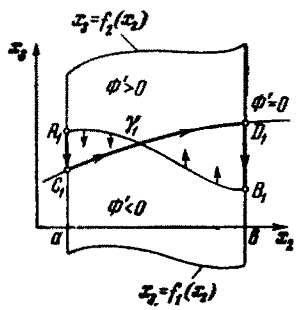

Fig. 7.

maximum and the minimum x_{1B} can be obtained. For example, if $\Phi' < 0$ below the curve $\Phi' = 0$, and if the problem is for the maximum x_{1B} we obtain, as is easily verified, the solution $A_1C_1D_1B_1$, represented in Fig.7.

It is easily seen that if the strip $a < x_2 < b$ is not bounded from above and below, only one of the two possible problems has a solution: the maximum problem, if $\Phi' < 0$ below the curve $\Phi' = 0$ (Fig.7), and the minimum problem, if $\Phi' > 0$ below this curve. Obviously, this result does not depend on the location of the points A_1 and B_1 on the straight lines $x_2 = a$ and $x_2 = b$.

6. In the case when the points A_1 and B_1 are both on the curve $\Phi'=0$,

this curve, obviously, represents the solution of the more simple problem for the degenerate functional and within the class of functions which do not possess corner points. For example, for the integral

$$J = \int_{\gamma_1} [-(x+y)^2\,dx + \sin \alpha y\,dy]$$

where γ_1 connects the points $A_1(0,0)$ and $B_1(1,-1)$, (Fig.8), we have

$$\Phi' = -\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 2(x+y)$$

Since $\Phi' < 0$ is below the curve $\Phi' = 0$, there exists the solution of the problem for the maximum J. It consists only of the portion of the curve $\Phi' = 0 (y = -x)$, since A_1 and B_1 are located on this curve. Let us verify, for example, that the straight line A_1B_1 is better than the curve $A_1C_1B_1$, which differs from it in the interval (0.1) by the variation $\delta y = kx$, where k = const. We obtain (Fig.8)

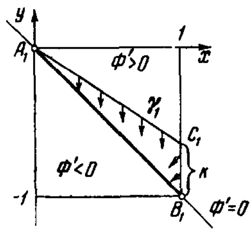

Fig. 8.

$$J[A_1B_1] = \int_0^{-1} \sin \alpha y\,dy$$

$$J[A_1C_1B_1] = \left(\int_{A_1}^{C_1}\right) + \left(\int_{C_1}^{B_1}\right) = \left(-\frac{k^2}{3} + \int_0^{-1+k} \sin \alpha y\,dy\right) +$$

$$+\left(0 + \int_{-1+k}^{-1} \sin \alpha y dy\right) = J[A_1B_1] - \frac{k^2}{3}$$

There results $\Delta J = -1/3k^2 < 0$, as was required.

We have thus corrected an error contained in the text [3], in the first as well as in the second editions, which consists in the statement that the functional J does possess neither a maximum nor a minimum for $0 < |\alpha| < \pi$. This error has arisen because, in calculating the increment ΔJ, the integral along the portion C_1B_1 was not considered, (Fig.8).

7. The representation of the variation in the form (2.1) is interesting because it is conserved also for an arbitrary number n of free

functions in Pfaff's equation (1.1). Thereby, the rank of the matrix H is equal to $n + 1$. An investigation of this case and its application to the problem of optimum rocket launching through choice of functions $\alpha(t)$ and $\mu(t)$ is easily carried out.

3. As an example, we solve the problem of maximum speed for equation (1.5) formulated in Section 1. We put

$$\delta t = 0, \quad x_1 = v, \quad x_2 = t, \quad x_3 = y$$
$$X_1 = \mu, \quad X_2 = v(\varphi - \mu p), \quad X_3 = \mu g, \quad \Phi' = -\Phi \tag{3.1}$$

From formulas (2.1) and (1.10) we obtain

$$\delta v = \int_0^T k(t)\,\Phi\,\delta y dt \qquad \left(\Phi = \mu g\left[\varphi + v\frac{\partial \varphi}{\partial v} - \frac{v^2}{g}\frac{\partial \varphi}{\partial y} - \mu p\right] = -\Phi'\right) \tag{3.2}$$

For simplicity, the law of resistance is approximated by a straight line. Then

$$c_{x0}v^2 = av - b, \qquad \varphi(v, y) = \frac{\rho_0 g F}{2G_0} H(y)(av - b) \tag{3.3}$$

(F is the area of the midship). Putting $H(y) = e^{(-ky)}$ and $P = \text{const} = P_0$, we obtain

$$\varphi(v, y) = \frac{g\varkappa}{\nu_0} e^{-ky}(av - b) \left(\varkappa = \frac{\rho_0 \nu_0}{2\beta}, \ \beta = \frac{G_0}{F}, \ \nu_0 = \frac{G_0}{P_0}\right)$$

Substituting $\phi\ (v, y)$, we obtain

$$\Phi(v, t, y) = \frac{\rho_0 g^2}{2\beta}\left\{\left[av + (av - b)\left(1 + \frac{kv^2}{g}\right)\right]e^{-ky} - \frac{1}{\varkappa}\right\}\mu \tag{3.4}$$

From the equation $\Phi = 0$ we see that for an arbitrary $v > 0$, there will always be one such value y, that the brace will be zero. As a

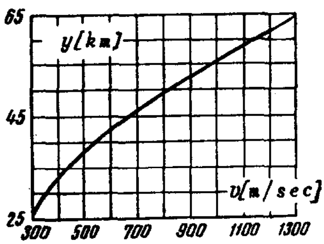

Fig. 9.

consequence, equation $\Phi = 0$ determines a real curve in the plane v, y (see Fig. 9), which corresponds to $a = 100$ m/sec, $b = 2 \times 10^4 \text{m}^2/\text{sec}^2$, $k \approx 10^{-4}\text{m}^{-1}$, $\beta = 10^2$ kg/m^2 and the value $\kappa^{-1} = 3300$ sec^2/m^2). Since $\mu(t) \neq 0$, the indicated curve does not depend on t and is determined exclusively by the adopted law of resistance, the law of decrease of density with altitude and the parameter κ, which depends on absolute constants and the construction parameters. If, considering g = const, we solve equation $\Phi = 0$ with respect to y, we obtain

$$y = \frac{1}{k} \ln \left\{ \left[av + (av - b)\left(1 + \frac{kv^2}{g}\right) \right] \varkappa \right\}$$

The different values κ of the point (v_0, y_0), corresponding to the initial value of the portion of the trajectory to be varied, may be situated in the general case either below the curve $\Phi = 0$, that is to say, in the region $\Phi > 0$ as well as above, in the region $\Phi < 0$.

Let us consider the motions (curves y_1) in the plane (ty) (Fig. 10). The curves $\Gamma_1'(A)$ and $\Gamma_1''(A)$ are drawn in this plane. They correspond to a vertical climb and a vertical fall, i.e., to most rapid change of velocity with time. All curves for other motions starting at point A are located between the curves Γ_1' and Γ_1''.

Let us consider first the case $\Phi(A) > 0$, that is $\Phi'(A) < 0$. In accordance with Section 2, the solution of the problem will leave point A along the left internal boundary $\Gamma_1'(A)$, that is, will start with a vertical climb. During the vertical climb, as may easily be verified, Φ decreases, i.e. the point $M(v, t, y)$ approaches the surface $\Phi = 0$. Assume that $\Phi(v_c, t_c, y_c) = 0$ at some point C. Inasmuch as on the plane ty the curve ϕ_1, which is determined by the point C and which satisfies the equations $\Phi = 0$ and (1.5), passes within the angle $\omega_1(C)$, the solution must pass, in accordance with Section 2, along the curve ϕ_1.

Depending upon the parameters T and y_k, the curve ϕ_1 may come to touch, as may be easily verified, either the boundary $y = y_k$, or the boundary $t = T$. In the first case, the point B_1 is on the internal boundary $\Gamma_1'(D)$, (where D is the point of contact with the boundary $y = y_k$ of the region of possible motion), and then the solution must terminate by the horizontal portion D_1B_1. Let us indicate this solution by $S_{v,h}$ (Fig. 10) corresponding to the vertical (v) start and horizontal (h) termination of motion. In the second case, the solution, in order to reach point B_1 in accordance with Section 2, must leave the curve ϕ_1 at such a point D_1 that the point B_1 can be reached along the internal boundary corresponding to this point. Such a boundary in the problem considered may obviously be only $\Gamma_1'(D)$. The solution obtained may be indicated analogously by $S_{v,v}$ (Fig. 10).

In the case when $\Phi(A) < 0$, that is $\Phi'(A) > 0$, the solution must leave the point A along the right internal boundary $\Gamma_1''(A)$, that is, must start by a horizontal flight. Thereby v increases and $y \equiv y_0$, such that Φ increases in accordance with (3.4). Let $\Phi = 0$ in some point C. Then, by analogy with the foregoing, the solution must again pass along the curve ϕ_1, determined by the point C and equations (1.5) and $\Phi = 0$.

The solution, obviously, may terminate again by either a horizontal or a vertical flight and, by analogy to the foregoing, we obtain the

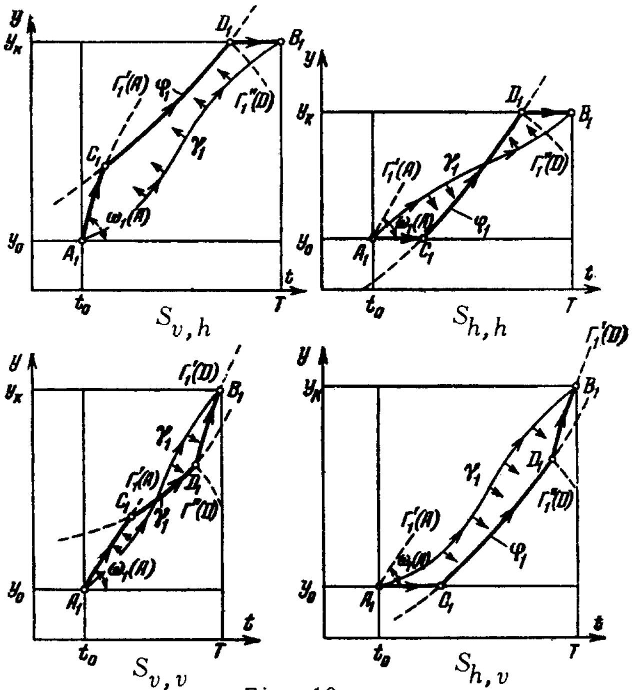

Fig. 10.

cases $S_{h,h}$ and $S_{h,v}$ for the horizontal start of motion (Fig. 10). The solution of the problem obtained, obviously, may be considered as being general. Through a change of the constants of the problem, the region of the motion along the curve ϕ_1 may be altered, and in the cases $S_{v,h}$ and $S_{h,v}$ may even vanish (when the points C_1 and D_1 coincide). The portions D_1B_1 and A_1C_1, obviously, may vanish in all four cases.

We note that under certain conditions (for example, if $b = 0, \rho \equiv \rho_0$), it may happen that the motion in accordance with the law $\Phi = 0$ becomes vertical at some point N_1 and becomes impossible any further (the curve ϕ_1 crosses the curve $\Gamma_1'(N)$). In this case, in accordance with Section 2, the solution must correspond to a vertical motion. If it leads to the intersection of the line $y = y_k$, it must change to a horizontal and terminate for $t = T$; and if it leads to the intersection of the straight line $t = T$, then it cannot correspond to a solution. The solution, in accordance with Section 2, will be terminated by vertical motion, beginning at some point D_1 preceding N_1, corresponding to reaching an altitude y_k for $t = T_1$.

The solution obtained is applicable, in accordance with Section 1, to the determination of optimum trajectory of free flight, and also to the optimum choice of the shape of the launching track of given altitude y_k for launching of a cosmic [space] rocket. For launching from a state of rest ($v_0 = 0$), as is seen from (3.3), we have $\Phi'(A) > 0$, for an arbitrary power law for drag and arbitrary thrust regime $P(t)$. This means that only cases $S_{h,v}$ and $S_{h,h}$ (Fig. 10) may be realized, that is, the motion will begin along a horizontal portion, changing at $v = v_c$ into a climb in accordance with the law $\Phi' = 0$.

If, in the formulation of the problem, the quantity y is not bounded from below, then, as is easily checked by the method of Section 2, the launching tracks, which descend first and then rise again, will be even more advantageous than launching tracks without a downward portion, while a vertical start will be relatively disadvantageous.

Remarks. 1. Inasmuch as for problems corresponding to the one considered (namely, the problem of maximum altitude $y_k = y(T)$ for prescribed terminal velocity and the problem of minimum time T for prescribed y_k and v_k), the equation $\Phi = 0$ and the boundary curves $\Gamma'(A)$ and $\Gamma''(A)$ (curves of vertical climb and fall) remain the same as in the problem considered, the corresponding optimum motions will consist of motions along the vertical and a motion in accordance with the law $\Phi(v, t, y) = 0$, and also, possibly, of horizontal motions at the ends, if the conditions of the problem so require (for the problem of min T this is confirmed by the recent paper by A. Miele [4]).

2. If θ is the angle between the velocity and a local horizon, the solution of the considered and the associated problems may be generalized almost without changes to the case of a central gravity field (merely the centrifugal term in (1.3) has to be added, which does not influence the resulting discussions and solutions).

3. In obtaining the solution, it was assumed, only for simplicity, that

$$c_{x0}v^2 = av - b, \qquad g = \mathrm{const}, \qquad \rho = \rho_0 e^{-ky}, \qquad P = \mathrm{const}\,.$$

For real laws of resistance and functions $g(y)$, $\rho(y)$ and $P(t)$, as well as taking into account counter pressure, the surface $\Phi(v, t, y) = 0$ resembles the one considered; the solution is obtained in the same fashion and yields results which are qualitatively analogous to the ones presented. However, the practical significance of the indicated example is not large. In Section 1, it was pointed out that in the idealized problems considered there, in contrast to real ones, the occurence of sharp turns in the trajectory and even of corner points is not connected with losses due to resistance. This explains why the solution has corner points, and this in turn diminishes considerably the practical significance of the example discussed, since in real problems concerning free flight and climb on a launching track, the decrease of the radius of curvature of the trajectory down to zero, as may be verified, leads to partial or complete loss of speed due to fast increase of resistance. Therefore, the solutions of idealized problems cannot be close to the solutions of real problems (the same can be said about the solution of A. Miele [4]).

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